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Geometrical Structure of Gravitation and Matter Fields

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Einstein's requirement of a unified geometrical description of gravitational fields and their matter sources is shown to become possible (at least for certain matter sources) by relaxing his other requirement of a minimal interaction of gravitation with matter. Arguments are presented to demonstrate that Schrödinger's discovery of pair creation by gravitational fields and the associated effects of virtual pairs make the relaxation of the latter requirement inevitable in order to obtain a complete macroscopic description (which needs no separate insertion to take account of averaged quantum effects). The gravitational field equations in case of a nonminimal interaction need higher derivatives of the metric than the second. The author's gauge theory on the manifold of the anti-de Sitter group $SO(3, 2)$ with the subgroup $SO(3, 1)$ (proper Lorentz group) as gauge group and the factor space of the two group manifolds as space-time manifold gives rise to a Yang–Mills field which can be interpreted to be composed of Riemannian curvature and a tensor formed out of torsion. Einstein's equations with a cosmological member are satisfied by the Cartan–Killing metric on the group manifold so that the generalization to a Kaluza–Klein theory results in a minimal disturbance of the group symmetry. The separation of the Yang–Mills field results in a part of its energy-momentum tensor becoming purely Riemannian; this part may be interpreted to be due to the contribution of virtual matter, whereas the part with torsion is due to real matter and its interaction with curvature. The Yang–Mills field equations have a third-order derivative purely metric part, which is equivalent to the field equations suggested by Yang (in the latter, however, torsion should be inseparably present and has been ignored). The torsion part is the "matter source" of this term and it is tempting to relate it to elementary particle spin. The theory can be regarded as a gauge theory of space-time geometry. It needs generalizations to geometrize matter with an energy-momentum tensor of nonvanishing trace. The equations, however, already considerably modify the problem of gravitational collapse. Further developments should serve to eliminate the "absurdity of relativity"—the collapse to a point (of which Einstein himself never became convinced).

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1. INTRODUCTION

The precision of the verification of Einstein's general theory of relativity has been steadily improving since its creation. In our solar system and—more qualitatively—in and even beyond our galaxy, the theory accounts best for the observed phenomena. The universal geometrical structure of the theory has convinced many physicists that its predictions ought to apply also to extreme conditions very remote from our experience. The creator of the theory himself had, however, already pointed out repeatedly in which respects he expected limitations to its validity and a necessity for its completion.

Einstein considered the right-hand member of his field equations as alien to the theory. He worked the rest of his life attempting the inclusion of the matter tensor into the geometrical structure.

A remarkable, today rarely cited paper (Einstein and Rosen, 1935) bears witness of his doubts. The paper suggests, from the example of the uniformly accelerated frame, modifications of the theory in which a geometrized source corresponding to elementary particle matter appears at the Schwarzschild radius of the spherically symmetric solution. The domain inside the horizon is eliminated in favor of a second sheet of the space outside.

The mathematical analysis of the solutions of the Einstein–Hilbert equations shows, however, that the gravitational collapse of macroscopic matter sufficiently close to the horizon should occur inevitably within a finite time period, as measured by a comoving observer. As soon as gravitating matter has passed the horizon its complete further collapse to a point is a mathematical consequence of the equations (Oppenheimer *et al.*, 1939).

During the same decade Schrödinger investigated solutions of Maxwell's and Dirac's equations in the metric of an expanding universe. He discovered in this connection the "alarming phenomenon" of the transition of the solution of an electron of negative energy into a state of positive energy, which he interpreted as pair creation caused by the gravitational field (Schrödinger, 1939, 1940).[2] A sophisticated aspect of quantum physics has thereby formally appeared in the classical theory.

A quantum field-theoretic treatment of gravitation was probably first suggested by Rosenfeld (1930) and it implied of course already the pair creation in gravitational fields. The success of quantum electrodynamics

[2]Remarkably, Schrödinger did interpret the analog of the effect for photons only as a reflection on the metric, not as pair creation. The pair creation of photons was stressed by the author (Halpern, 1962).

inspired many authors to continue the development started by Rosenfeld. De Witt (1952) approached the subject in a fully covariant form, whereas Feynman (1971; and numerous previous and subsequent presentations since 1952) Gupta (1952), Thirring (1961), Tonnelat, and others began to develop a spin-two theory in flat space which started out from the linear Fierz–Pauli equation and a special form of Einstein's equations found by Papapetrou (1954) in which the nonlinear term is identified with the symmetrized energy-momentum tensor of the gravitational field which acts as its own source to the linear Fierz–Pauli term. The gauge invariance of the linearized theory was conjectured to become in all orders the covariance of Einstein's theory (Kraichnan, 1955; Gupta, 1959; Halpern 1963*a*,*b*; Deser, 1970).

The author later remarked (Halpern, 1963*a*,*b*) that the complementarity of geometry and physics in the description of nature, discovered already by B. Riemann in the last century, made the manifestly covariant approach largely equivalent to the perturbative flat space approach from linear field theory to all orders. Both approaches lead in higher orders to unresolved divergence problems. In finite order these can be renormalized by introduction of counter terms into the Lagrangian of the gravitational field which are nonlinear in the curvature (Utiyama, 1962; De Witt and Utiyama, 1962)

$$\mathscr{L} = \sqrt{g}\,(R + aR^2 + bR_{iklm}R^{iklm}) \tag{1}$$

Yet, even if the renormalization problem is solved completely, the magnitude of the nonlinear admixture remains in the dark because it depends on contributions from all possible elementary particle fields, including the self-interacting gravitational field.

All the vacuum solutions of the Einstein–Hilbert equations are also solutions of the modified equations resulting from the Lagrangian of equation (1); other physical solutions of the latter are not known.

Schrödinger, whose assistant I was during his last years, was well aware of the prediction of gravitational collapse by general relativity (he first used the comparison with Xeno's paradox of Achilles and the turtle). His great experience in the development of physics and also his "alarming phenomenon" made him, however, doubt the validity of the theory in all its limits (Halpern, 1987). His views largely agreed with those of Einstein. The author was influenced by these views to search for a different way out and thus began to examine the modifications of the classical gravitational field equations by virtual elementary particle processes in their effects on the test case of a spherically symmetric mass distribution (Halpern, 1967, 1971). Many physicists believed that a collapse through the horizon, although predicted by the Einstein–Hilbert equations, would practically never occur because the initial conditions which guarantee it to happen are too restrictive as to be of practical importance.

The author's view is, however, that even if this *could* occur with a set of measure zero of all the possible initial conditions a problem for the theory exists that asks for a new outlook. The difficulties of finding approximate alternative solutions of the modified equations proved, however, forbidding.

Some time after the author's work, Sakharov (1967) suggested an approach to obtain similar modified equations. Both the earlier attempt of the present author and the independent one by Sakharov are related to an idea of O. Klein, according to which the Lagrangian of general relativity is to be obtained from the quantum fluctuations of spinor elementary particle fields; also a work by Weyl (1929) foreshadows such ideas.

The concentration on the virtual quantum effects of the gravitational interaction with pairs happened because the real pair formation, known to everyone who somewhat understood the subject since the works of Rosenfeld and Schrödinger, appeared trivial and less important.[3]

The author treated it earlier as one of the simplest elementary particle processes, mainly to demonstrate how and to what degree elementary particle processes can at all be caused by the gravitational field in spite of the principal of equivalence (Halpern, 1962). Common knowledge included of course that the total Schwarzschild metric cannot be static—but the author, influenced by Schrödinger's insistence, considered the domain beyond the horizon as unphysical—else one might even argue that already Schrödinger's early work foresaw the Hawking radiation, because also the de Sitter universe, discussed extensively by Schrödinger, has a horizon (Schrödinger, 1957). It appeared already clear that collapsing matter interacting only gravitationally should in general give rise to some pair creation, the likelihood of the process increasing, the closer this matter comes to the formation of a horizon. This should happen in analogy to the situation in nongravitational elementary particle physics, roughly, because the collapsing matter can transmit gravitationally momentum to virtual pairs and supply the lacking balance to make it real. The formation of a horizon is not necessary for this process.

However one looks at the situation, one will have to agree that in order to take account of the average effects due to elementary particle pairs in the large, one needs a modification of the classical gravitational field equations. The Einstein–Hilbert equations do not forbid pair creation, but

[3]When Iwanenko at the GR2 meeting in 1962 in Poland urged Feynman to give a written acknowledgment of such a process, calculated by his pupil Vladimirov, Feynman wrote jokingly that he agreed that the reaction may occur once in the universe.

they can certainly not derive or predict it even in average. The argument that the Einstein–Hilbert equations are the correct equations everywhere and need only the insertion of the effect of elementary particle pairs by hand may prove physically as wrong as the well-known historical error that the equations of classical mechanics are the correct ones in all domains and need only insertion by hand of statistical thermal fluctuations.

We thus adopt here the view that:

1. Schrödinger's discovery of pair creation by metric fields leads directly to the requirement of modifications of the macroscopic gravitational field equations which should only in extreme or very special situations lead to significantly different physical results than the Einstein–Hilbert equations.
2. These equations have derivatives of higher order than the second and the interaction of matter with the gravitational field is not restricted to be minimal; it includes the curvature to take account of averaged contributions by virtual or real particle pairs.

We would like to stress here once more a point raised before: We do not claim that real pair creation necessarily plays tacitly a significant role in the physics of the universe—but it is not predicted by the Einstein–Hilbert equations; the existence of the process, however small, allows the conclusion (for which we have otherwise not enough evidence) that also the effect of virtual pairs is not included. We know from quantum electrodynamics about the great significance of these virtual particle effects for the law of motion. We can expect that these are more significant on longer distances in the case of the universal gravitational interaction which deals also with virtual pairs of massless particles.

We have of course good reasons to trust general relativity for the conditions in our solar system and indications that it still applies well for neutron stars. The history of physics demonstrates that laws well established in a certain domain had to be modified in different domains and one should not neglect indications however faint for it. This can be well understood because the mathematical description is based on axioms which introduce idealizations with which a measurable physical object is identifiable only to an approximation which is bound to fail in more extreme situations. This was expressed in the author's presentation at the Wroclaw symposium (Halpern, 1993) as: "Every good and therefore clear theory can be expected to reveal ultimately so great an absurdity that no reasonable person can believe in it."[4] Indeed this proved to be the case not only for

[4]A poster with the same content was presented on July 3, 1992, at the GR13 meeting in Cordoba.

Newtonian mechanics, but also with Maxwellian electromagnetic theory, Rutherford's atom, and classical statistical mechanics. The assumption that it will not apply finally also to the theory of relativity is not convincing to the author. The author adopted tentatively the point of view that the prediction of the collapse of a large cloud of dust to a point irrespective of the interaction of the dust particles has the character of such an absurdity. Its elimination promises to reveal valuable new aspects.[5]

The search for modifications of the gravitational field equations which avoid the absurdity and take account of the discussed elementary predictions of quantum field theory (at least in average for classical fields) does not need to be restricted to terms derived from equation (1). These were obtained from an early approach to quantum theory in gravitational fields. The assumptions about the gravitational law in the quantum domain is indeed based on speculations derived from the macroscopic law and the quantum laws of elementary particle fields. Even the macroscopic quantum effect, which can be expected in analogy to the Casimir effect of the electromagnetic interaction, remains remote from any observation (Halpern, 1987). Schrödinger realized that we have no evidence for quantum effects including gravitation and suggested in his lectures the search for an extended gravitational law that would give rise at small distances to mesonic forces (Halpern, 1987).

The search for a gauge theory of gravitation also appears promising. The first attempt was probably made by Utiyama (1956). A somewhat different approach was then suggested by Kibble (1960). The formulation in terms of principal fiber bundles was given by Cho (1975, 1976).

The cited authors strive and claim to obtain the Einstein–Hilbert equations as gravitational field equations. This results from the arbitrariness in the choice of a Lagrangian and the variation after the gauge field is determined. The present author in his earlier gauge field approach, following Klein's suggestion, used the group of spin transformations (locally isomorphic to the Lorentz group) to obtain an admixture of a nonlinear Lagrangian equation (1). After the Lagrangian was obtained, the variation was performed with respect to the metric or the tetrad. This results in an admixture of terms with fourth-order derivatives to the field equations (Halpern, 1987). Yang (1974) considered a gauge theory of gravitation with the general linear group $GL(4, r)$ as gauge group. He presented metrical field equations of the form

$$R_{ij;k} - R_{ki;j} = 0 \tag{2a}$$

[5]It must be remarked here that the 3°K background radiation lends support to the assumption that the universe evolved from a very limited domain; the conclusion about the contraction of a cloud of dust to a point seems, however, too far-fetched.

and stated that the second-order metric equations, especially the Einstein–Hilbert equations, were incorrectly derived from a gauge theory.

Equation (2a) admits obviously all vacuum solutions of the Einstein equations, but it was soon recognized that it admits besides this also other solutions of unphysical character (Pavell, 1975).

The present author transformed Yang's equations with the Bianchi identities into the form

$$R^h{}_{ijk;h} = 0 \tag{2b}$$

This form shows that they are just the Riemannian analog of Maxwell's equations, if one expresses them in terms of the curvature two-form.

The author, however cannot agree that a gauge principle with $GL(4, r)$ or its subgroups would result in the Riemannian curvature alone—except for very special solutions. The curvature turns out to be in general non-Riemannian with nonvanishing torsion. What should exclude the torsion in Yang's approach?

The author observed later that the principal fiber bundle $P(G, H, G/H, \pi)$ of a Lie group G with a Lie subgroup H and the coset space G/H with the natural projection $\pi: G \to G/H$ forms a gauge theory on the base space $B = G/H$, which for the manifold of certain semisimple groups allows one even to formulate a realistic Kaluza–Klein type-metric theory [as a simple case: $G = SO(3, 2)$ and $H = SO(3, 1)$, the anti-de Sitter group and the proper Lorentz group with the anti-de Sitter universe as base G/H].

The natural Cartan–Killing metric is namely for every semisimple group G_r a solution of the Einstein–Hilbert equations in r dimensions with cosmological member. Restricting other solutions such that the Killing vectors corresponding to the left invariant vectors of H and their commutation relations are preserved, one arrives at a Kaluza–Klein theory with a metric on the base manifold which is the projection of the solution on the bundle space P. The gauge group $SO(3, 1)$ is a subgroup of $GL(4, r)$, so that the gauge field is related to that of $GL(4, r)$—the curvature is, however, even in this case non-Riemannian. Torsion is usually present. A fundamental difference from the approach by Yang, is however, use of a pseudo-orthogonal subgroup of $GL(4, r)$ so that the theory is metric (it determines a covariant derivative which, applied to the metric, vanishes). The connection can in such a case be separated into a Riemannian part and a contortion part.

The (homogeneous) Einstein–Hilbert equations projected on the base have thus in general a purely metric term, a term formed only out of the torsion tensor and its covariant derivatives, and a mixed term formed of both the Riemannian curvature as well as the torsion. The author restricted in most of the publications on this theory attention to the special solutions for which torsion vanishes; this was somewhat under the influence of a view of Dirac.[6]

A brief discussion with Dr. A. Lasenby, who expressed a general opposing view after my Wroclaw lecture, encouraged me to reconsider the torsion problem.

The structure of the modified equations of the author's gauge theory appears already interesting enough in the special case of vanishing torsion. They can be obtained by writing down the equations of the Kaluza–Klein gauge theory for a general curvature two-form and choosing then the special cases of a purely Riemannian curvature (Halpern, 1993). The results are one set of equations consisting only of the Yang term, (2a), and a second set consisting of the Einstein–Hilbert term and a term bilinear in the curvature, which may be called the energy-momentum tensor of the gauge field. The latter expressions will become of the correct physical dimension if multiplied by a constant with the dimension of length squared—the square of the Planck length—to yield the expected magnitude of the effect.

The theory does not answer the question of the origin of this magnitude. The other unit of length, which is defined by the theory, is that of the radius of the de Sitter universe. Empirically the dimensionless ratio of these two length is of the order of 10^{65} (about Dirac's large number to the power 3/2) (Dirac, 1937); the second length determines the cosmological member of the field equations [dimension $(\text{length})^{-2}$].

One may introduce by hand a right-hand member as source to these torsion-free field equations. The second set obviously will have the energy-momentum tensor of matter as source. The Yang term may have a source that is related to elementary particle spin. Such a source in general will be averaged out of macroscopic matter at higher temperature. In the latter case the energy-momentum tensor of matter will be covariantly conserved; in general the conservation law requires, however, an additional term which is due to the interaction of the curvature tensor with the spin source of the Yang term—the gravitational analog of a nonminimal Pauli interaction term.

The theory restricted to these special solutions is indeed a peculiar Kaluza–Klein type theory in which the analog of the electromagnetic field is the Riemannian curvature two-form of the metric and the "charge" is related to the elementary particle spin. The typical fiber of the bundle is in this simplest version of the theory the proper Lorentz group $SO(3, 1)$. It

[6]When asked about his view on torsion in gravitational theory, Dirac replied that he considered it and came to the conclusion that it is so peculiar a geometrical feature "that it probably will have no place in physical theory."

may be identified (or associated) with the bundle of orthonormal frames. Elementary particles with a spin are described by a wave function which is a functional realization (and not a matrix representation) of the group; this is, however, in most cases equivalent to a matrix representation with a wave function of space-time. Half-integer spin can remarkably also be described by such functional realizations with the parameters of the group manifold as variables (Bopp and Haag, 1959). A more sophisticated form of the theory deals with the universal covering group of $SO(3, 2)$ as G and its six-dimensional subgroup as H. Remarkable topological features relate it to particle quantization and spin statistics. We are restricting ourselves here to $G = SO(3, 2)$. We assume the *homogeneous* Einstein equations in ten dimensions as the field equations of our $(6+4)$-dimensional gauge theory on the manifold of $SO(3, 2)$.

The curvature two-form (gauge field) is in general not Riemannian, but in our case of an orthogonal subgroup of the general linear group as gauge group one can decompose it into a Riemannian curvature two-form and a term with torsion (contortion). One can split the total field equations such that the purely Riemannian terms are on the left-hand side and the mixed and pure torsion terms on the right-hand side. The horizontal component of the field equations consist of the Einstein–Hilbert term plus a term bilinear in the Riemann tensor (additional energy-momentum density of the gravitational field). The mixed horizontal–vertical term is the Yang term and the purely vertical term is eliminated with a Lagrange multiplier guaranteeing on the fibers the original group metric. The right-hand side of the field equations containing torsion is interpreted as the matter tensor comprising a nonminimal interaction term of matter with the Riemannian curvature.

The geometrical features of the theory are discussed in Section 2. The resulting geometry of space-time turns out to be related to the Einstein–Cartan manifold, used extensively for newer developments of Cartan's original idea (Hehl *et al.*, 1976). The structure of the equations and their physical interpretation are however, very different. Hehl's theory is dualistic. The arena of space-time exists independently of the "animals" presented to the public by the physicists who strive to create the illusion of showing nature. The present theory is unitarian. The arena itself is alive and comprises about everything in the universe—except for the initial conditions. The gauge theory is that of the geometry and even its field equations are inherent in the structure of the latter.

2. THE MATHEMATICAL STRUCTURE OF THE THEORY

The field equations of our theory are derived from a gauge principle which is inherent in the structure of the principal fiber bundle:

$$P(G, H, G/H, \pi) \tag{3}$$

with G a semisimple Lie group and H its semisimple Lie subgroup, which is isomorphic to the typical fiber. The space of left cosets G/H forms the base manifold and π is the natural projection: $G \to G/H$.

We choose here for G the anti-de Sitter group $SO(3, 2)$ and for H the proper Lorentz group $SO(3, 1)$. [A choice of the universal covering group of $SO(3, 2)$ for G and the corresponding subgroup H leads to interesting topological features which pertain to elementary particle quantization and spin-statistics and are discussed in other publications.] We can choose left invariant vectors A_R which are at every point of the manifold of G orthonormal with respect to the natural Cartan–Killing metric γ:

$$\gamma_{RS} = \operatorname{tr}(Ad'(A_R)Ad'(A_S)) = c^U_{RV}c^V_{SU} = \mp\delta_{RS} \tag{4}$$

with $c^U_{RV} = [A_R, A_V]^U$ the structure constants of G.

We can in particular choose four of the orthonormal vectors A_E $(E = 1, \ldots, 4)$ perpendicular to the fiber and the remaining six A_M $(M = 5, \ldots, 10)$ on the fiber through each point. We shall henceforth always denote all components of tensors with respect to such a base by letters $A, \ldots, L$ if they are perpendicular to the fiber and by letters $M, \ldots, Q$ if they are vertical (on the fiber); general components are denoted by letters $R, \ldots, Z$. We apply this rule even without further warning to the Einstein summation convention: A_EB^E thus sums over one to four, A_MB^M sums over five to ten, and A_RB^R over one to ten. We adopt the same rules for the coordinates of local trivialization, for which, however, lower case indices are used.

The natural metric γ of a semisimple Lie group of r dimensions always fulfills Einstein's equations with a cosmological member:

$$R_{UV} - \frac{1}{2}\gamma_{VU}R + \gamma_{UV}\frac{r-2}{8} = 0 \tag{5}$$

The metric γ can be projected by π on the base manifold G/H and yields there a metric $g = \pi'\gamma$. The base manifold becomes thus a homogeneous space of constant curvature, which in our case is 4-dimensional and has the topology and metric of the anti-de Sitter universe. The metric g then fulfills Einstein's equations with a different cosmological member: Denoting the Ricci tensor and the curvature invariant of g by B_{ik} and B, we obtain

$$B_{ik} - \frac{1}{2}g_{ik}B + \frac{1}{2}g_{ik} = 0 \tag{5a}$$

The cosmological member in our units has the value $\Lambda = 1/2$ and the radius of the anti-de Sitter universe $\rho = (6)^{1/2}$.

We shall consider other solutions γ of equation (5) besides that which results in the metric of the de Sitter universe. Such solutions will have larger local curvature. For example, due to gravitational waves, these give rise to other lengths besides ρ. More complicated solutions give rise to torsion on the base manifold. We shall relate torsion to matter and we shall need then another fundamental length to relate the intensity of torsion with the curvature. We know empirically that this must be the Planck length $(\hbar G/c^3)^{1/2}$, which in units with $\hbar = c = 1$ is the square root of Newton's G. We choose here largely out of convenience the anti-de Sitter universe with its radius ρ unrelated to $\sqrt{G}$—namely to obtain the principal fiber bundle P on a group manifold. Theories with G a variable scalar field which determines ρ by second-order field equations were already conceived by Einstein and later worked out (on the basis of Dirac's large numbers hypothesis) by Jordan (1959), Thiry (1948), and Brans and Dicke (1961). We can obtain related features by choosing $G = SO(4, 2)$ and $H = SO(4, 1)$ with a five-dimensional base manifold which allows us to construct a Kaluza–Klein–Jordan theory for electromagnetic fields.

The most general admissible solution of equation (5) is a metric γ which still has six orthonormal Killing vector fields A_M which lie on the fiber through every point of the manifold of G and have the commutation relations of the Lie subalgebra of H. There exist also four orthonormal vector fields A_E which are everywhere perpendicular to the A_M with respect to the metric γ and these also commute with the Killing vectors A_M as prescribed by G:

$$[A_M, A_N] = c^P{}_{MN} A_P, \qquad [A_E, A_M] = c^F{}_{EM} A_F \tag{6a}$$

Only the commutation relations between the A_E are generalized:

$$[A_E, A_F] = \mathscr{C}^R{}_{EF} A_R \tag{6b}$$

where in general $\mathscr{C}^R{}_{EF}$ are functions of the points of the base and not equal to the $c^R{}_{EF}$. The fiber bundle structure is still the same and it determines a more general metric $g = \pi'\gamma$ on the base manifold. The topology of the manifolds is unchanged. A connection on P is given by choosing the horizontal vector spaces perpendicular with respect to γ to the vertical vector spaces spanned by the A_M at every point of G. The horizontal vector space is thus spanned by the A_E. The corresponding one-forms are denoted by A^E, A^M.

A local trivialization defines coordinates x^r: x^e, x^m, on G such that the first four x^e label the points on an open set of the base and can thus also be used as coordinates for this base domain. We shall henceforth make use of this without further mention. An orthonormal base of horizontal vectors A_E at any point of G projects on vectors $\pi' A_E = e_E$, which are orthonormal with respect to g on the base. The projection of the frames of the vector fields A_E for all points of the fiber over a base point results in the bundle of orthonormal frames of $SO(3, 1)$ on the base point and vice versa. This frame bundle can thus be identified with the fiber over base points, which is frequently done in the mathematical literature for subgroups of the general linear group. We shall also make use of it to obtain torsion and curvature on the base. The Lie algebra-valued connection one-form is

$$\omega = A^M \hat{a}_M \tag{7}$$

where $\hat{a}_M$ are elements of the Lie algebra corresponding to A_M. Using instead of the abstract $SO(3, 1)$ its bundle of orthonormal frames, we can substitute

$$\omega = A^M c^E{}_{MH} \tag{7a}$$

with the matrix representation of the generators $\hat{a}_M$ equal to the structure constants $c^E{}_{MH}$ as determined by the frame mapped from the base on that surface of each fiber H where the origin of the group coordinates x^m is chosen. The curvature two-form becomes

$$\Omega = D\omega = d\omega + [\omega, \omega] = d\omega + \frac{1}{2} c^P{}_{MN} A^M \wedge A^N \hat{a}_P \tag{7b}$$

or, for horizontal frame vectors,

$$\Omega(A_E, A_H) = \mathscr{C}^M{}_{HE} \hat{a}_M \tag{7c}$$

The connection ω is linear and the soldering form θ ($\mathscr{R}^4$-valued one form) in a local trivialization has the components

$$\theta^H(p) = A^H(p) \qquad (p \in P,\ H = 1, \ldots, 4) \tag{8}$$

This is so because the vectors $\pi' A_H$ on the base actually do form the frame bundle corresponding to every point p of G.

The torsion two-form becomes

$$\Theta = D\theta = d\theta + [\omega, \theta] \tag{8a}$$

or for horizontal frame vectors

$$\Theta(A_E, A_F) = \mathscr{C}^H{}_{FE} \qquad (H + 1, \ldots, 4) \tag{8b}$$

To the above horizontal forms on P there correspond uniquely tensors on the base. Let us consider these forms at the point $p \in P$ to which there corresponds a frame of the bundle of orthonormal frames given by the base vectors: $e_J = \pi' A_J(p)$. The components of the tensors on the base in this orthonormal frame are δ^I_J for the soldering form θ, $T^H{}_{IJ} = \mathscr{C}^H{}_{JI}$ the torsion tensor, and $H^A{}_{EIJ} = \mathscr{C}^P{}_{JI} c^A{}_{PE}$ the curvature tensor.

The Lie algebra-valued connection one-form ω is vertical. The pullback of any local section s on P defines a gauge potential $A = s^*\omega$ and the pullback of the curvature two-form a gauge field: $F = s^*\Omega$. This gauge field is related in the given frame to the curvature tensor H just given before.

The gauge potential depends on the local section, which may again be canonically associated to a local trivialization ϕ. The gauge transformation of the potential is thus related to a base-point-dependent transformation of the vertical coordinates x^m by the action of a group element $g(x)$:

$$x^m = \varphi^m(g(x'^e), x'^m), \qquad x^b = x'^b \tag{9}$$

and the transformation of the coordinate components A^M_m of the one-form A^M, $\hat{a}_M$, resulting from it. It can be expressed in terms of the gauge potential alone with the help of the Maurer–Cartan form Θ_{MC} on the fiber (in coordinates: $A^M_m \hat{a}_M$) as

$$A'(x^e) = Ad(g^{-1}(x^e))A + g^*\Theta_{\mathrm{MC}} \tag{10}$$

The last inhomogeneous term contains in fact the vertical coordinate components of $A^M \hat{a}_M$ as they appear in the coordinate transformation: $A^M_m \, \partial x^m/\partial x'^e$. The picture of a (local) transformation of the vertical coordinates caused by a base-point-dependent group element $g(x^e)$: $x^m = \varphi^m(x'^n, g(x^e))$ resulting in a transformation of

$$\begin{aligned} A^M_i \hat{a}_M &= A'^M_r \frac{\partial x'^r}{\partial x^i} \hat{a}_M = \left(A'^M_e \frac{\partial x'^e}{\partial x^i} + A'^M_m \frac{\partial x'^m}{\partial x^i} \right) \hat{a}_M \\ &= \left(A'^M_i(x') + A'^M_m(x') \frac{\partial x'^m}{\partial x^i} \right) \hat{a}_M \\ &= Ad(g^{-1}) A'^M_i(x) \hat{a}_M + A'^M_m(x'^n) \frac{\partial x'^m}{\partial x^i} \hat{a}_M \end{aligned} \tag{10a}$$

which is used in the Kaluza–Klein form of the theory, is more transparent for a reader accustomed to the relativistic formalism. Also here the value of ω at two different surfaces (determined by $x^m = 0$ and $x'^m = 0$) enter into the description.

The bundle of orthonormal frames results in a metric connection for which the covariant derivative of the metric tensor vanishes (this is not the case for the general frame bundle). We are here able to decompose the linear connection $\bar{\omega}$ on the base into the Riemann connection of the metric g and the contortion tensor K formed out of the torsion tensor T:

$$K^i{}_{jk} = \frac{1}{2}(T^i{}_{jk} + T_{kj}{}^i + T_{jk}{}^i) \tag{11}$$

To show this, one expresses the torsion two-form in our orthonormal frame as a commutator and decomposes it formally as the commutator of the orthonormal frame projected on the base plus components of the connection $\bar{\omega}$:

$$\begin{aligned}[A_E, A_F]^H &= [e_E, e_F]^H + c^H{}_{ME} A_i^M A_F^i - c^H{}_{MF} A_i^M A_E^i \\ &= [e_E, e_F]^H + \Gamma^H{}_{FE} - \Gamma^H{}_{EF} = T^H{}_{FE} \end{aligned} \tag{12}$$

The Riemann connection in an orthonormal frame is expressible in terms of commutators. One finds thus

$$\Gamma^H{}_{EF} = \left\{ \begin{matrix} \overline{H} \\ EF \end{matrix} \right\} - K^H{}_{FE} \tag{12a}$$

with the components of the Riemann connection of the metric g denoted by

$$\left\{ \begin{matrix} \overline{H} \\ EF \end{matrix} \right\}$$

and $\Gamma^H{}_{EF}$ the complete connection on the base which can be split into Riemann connection and contortion. Such a separation is not possible with the full general linear group as gauge group.

The expression of T as a commutator implies, due to the Jacobi relations,

$$T^J{}_{AJ} = K^J{}_{AJ} = 0 \tag{13}$$

Several authors, notably C. N. Yang, derived field equations with a curvature tensor from the full linear group $GL(4, r)$, equation (2a). They assume that their curvature is Riemann, but we conclude that it is a general curvature, which does not even allow a separation into metric and torsion terms as it occurs above equation (12a).

The Riemannian curvature tensor on G, expressed in terms of the Riemann connection and its derivatives in an orthonormal frame A_R, has the components

$$\begin{aligned} R^X{}_{UVW} &= \left\{ \begin{matrix} X \\ WU \end{matrix} \right\}_{|V} - \left\{ \begin{matrix} X \\ VU \end{matrix} \right\}_{|W} + \left\{ \begin{matrix} X \\ VZ \end{matrix} \right\} \left\{ \begin{matrix} Z \\ WU \end{matrix} \right\} \\ &\quad - \left\{ \begin{matrix} X \\ WZ \end{matrix} \right\} \left\{ \begin{matrix} Z \\ VU \end{matrix} \right\} - \mathscr{C}^Z{}_{VW} \left\{ \begin{matrix} X \\ ZU \end{matrix} \right\} \end{aligned} \tag{14}$$

The Riemann brackets in such a frame can be expressed in terms of the commutators of the frame vectors A_R:

$$\left\{ \begin{matrix} R \\ ST \end{matrix} \right\} = \frac{1}{2} [\mathscr{C}^R{}_{ST} + \gamma^{RU} (\gamma_{SV} \mathscr{C}^V{}_{UT} + \gamma_{TV} \mathscr{C}^V{}_{ST})] \tag{14a}$$

Each of the commutators $\mathscr{C}^R{}_{ST}$ is in our geometry either a structure constant of G or a component of the curvature or torsion two-forms.

Vertical derivatives of each of these commutators vanish because of the commutation relations (6a), (6b) and the Jacobi relations; their invariant derivatives, are therefore in this frame equal to ordinary derivatives. They become on the base covariant derivatives with the connection Γ, equation (12a). The whole expression is a tensor on G and therefore covariant with respect to rotations of the frame, and also the expression on the base which results from a projection must be covariant with respect to the corresponding rotations of the projected frame and therefore a tensorial expression.

The purely vertical components of R are expressible by the structure constants of the subgroup H alone:

$$R^O{}_{MNP} = \frac{1}{4} c^O{}_{MQ} c^Q{}_{NP} \tag{14b}$$

$$R^E{}_{MNP} = O \tag{14c}$$

The other components are

$$R^M{}_{EIN} = \frac{1}{4}(\gamma_{NP}\gamma^{JA}\mathscr{C}^M{}_{IA}\mathscr{C}^P{}_{JE} + c^M{}_{NQ}\mathscr{C}^Q{}_{IE}) \tag{14d}$$

$$R^M{}_{NEI} = c^M{}_{NQ}\mathscr{C}^M{}_{EI} + \frac{1}{4}(\mathscr{C}^M{}_{EJ}\mathscr{C}^P{}_{BI} - \mathscr{C}^M{}_{BI}\mathscr{C}^P{}_{JE})\gamma^{BJ}\gamma_{PN} \tag{14e}$$

$$\begin{aligned} R^A{}_{NIJ} = {} & \frac{1}{2}\gamma_{NP}\gamma^{AF}(\mathscr{C}^P{}_{EJ|I} - \mathscr{C}^P{}_{EI|J}) \\ & + \frac{1}{2}(\mathscr{C}^P{}_{BJ}K^A{}_{KI} - \mathscr{C}^P{}_{BI}K^A{}_{KJ} + \mathscr{C}^{PA}{}_K T^K{}_{IJ})\gamma^{BK}\gamma_{PN} \end{aligned} \tag{14f}$$

$$\begin{aligned} R^A{}_{EIJ} = {} & \frac{1}{4}\gamma_{PQ}\gamma^{AH}(\mathscr{C}^P{}_{HI}\mathscr{C}^Q{}_{JE} - \mathscr{C}^P{}_{HJ}\mathscr{C}^Q{}_{IE} - 2\mathscr{C}^P{}_{IJ}\mathscr{C}^Q{}_{HE}) \\ & - c^A{}_{PE}\mathscr{C}^P{}_{IJ} + K^A{}_{EJ|I} - K^A{}_{EI|J} + K^A{}_{HI}K^H{}_{EJ} \\ & - K^A{}_{HJ}K^H{}_{EI} - T^H{}_{JI}K^A{}_{EH} \end{aligned} \tag{14g}$$

In (14f) and (14g) the torsion and contortion tensors T and K have replaced the commutators to save space. Considering the relation

$$\mathscr{C}^P{}_{AE}\mathscr{C}^Q{}_{IJ}\gamma_{PQ} = -3\mathscr{C}^P{}_{AE}c^H{}_{PK}\mathscr{C}^Q{}_{IJ}c^K{}_{QH} = 3H^H{}_{KAE}H^K{}_{HIJ} \tag{15}$$

following from (6a), (6b), the components of R are expressed in terms of the curvature and torsion tensors on the base. The fact that the horizontal derivatives (denoted by a bar) of the commutators become covariant derivatives with the connection of equation (12a) on the base has already been discussed.

The tensor H_{AWIJ} can be decomposed into the Riemann tensor on the base and a tensor formed solely out of the contortion tensor and its derivative with the Riemann connection of the metric g (denoted here by a semicolon):

$$\begin{aligned} H_{AEIJ} &= B_{AEIJ} + Q_{AEIJ} \\ Q_{AEIJ} &= K_{AEI;J} - K_{AEJ;I} + K_{AHI}K^{H}{}_{EJ} - K_{AHJ}K^{H}{}_{EI} \end{aligned} \tag{16}$$

The Riemannian curvature tensor on G can thus be expressed on the base into the Riemannian curvature tensors and their first covariant derivatives on the base with the projected metric g, plus expressions formed out of the contortion tensor and its first covariant derivative. All these covariant derivatives are with the Riemann connection formed out of g. The expressions are at most linear in the first covariant derivative of the Riemann tensor, bilinear in the Riemann tensor, bilinear in the first covariant derivative of the contortion tensor, and quadrilinear in this tensor.

We can infer from Jacobi's relations applied to

$$c^{J}{}_{PE}\mathscr{C}^{P}{}_{IJ} = -H^{J}{}_{EIJ} = -(B_{EI} + Q^{J}{}_{EIJ})$$

that

$$H^{J}{}_{EIJ} = H^{J}{}_{IEJ} \tag{16a}$$

and

$$Q^{J}{}_{EIJ} = Q^{J}{}_{IEJ}$$

3. THE FIELD EQUATIONS

The field equations of our theory are the *homogeneous* Einstein equations in ten dimensions with a cosmological member. The solutions are subject to the restrictions imposed by the vertical Killing vectors. These restrictions will, as we shall see, largely eliminate the independent character of the vertical components of the field equations. We showed in the last section that all the components of the curvature tensor on P are expressible by tensorial terms which depend only on the points of the base. We obtain

$$R_{MN} = \frac{1}{6}\gamma_{MN} - \frac{1}{4}\mathscr{C}_{M}{}^{IJ}\mathscr{C}_{NIJ} \tag{17a}$$

$$R_{MI} = \frac{1}{2}\mathscr{C}_{MI}{}^{J}{}_{|J} + \frac{1}{4}\mathscr{C}_{MJK}T_{I}{}^{JK} \tag{17b}$$

$$R_{EI} = c^J{}_{PE}\mathscr{C}^P{}_{JI} + \frac{1}{2}\mathscr{C}^P{}_{JE}\mathscr{C}_P{}^{JI}$$
$$- K^J{}_{EI|J} + K^J{}_{HI}K^H{}_{EJ} - T^H{}_{JI}K^J{}_{EH} \tag{17c}$$
$$R = 1 + \frac{1}{4}\mathscr{C}^P{}_{JI}\mathscr{C}_P{}^{JI} + K^J{}_{HI}K^H{}_{IJ}$$
$$- T^{HI}{}_J K^J{}_{IH} + c^J{}_{PI}\mathscr{C}^P{}_J{}^I \tag{17d}$$

The field equations

$$G_{UV} = R_{UV} - \frac{1}{2}\gamma_{UV}R + \gamma_{UV} = 0$$

are

$$G_{MN} = \frac{1}{4}\mathscr{C}_M{}^{IJ}\mathscr{C}_{NIJ} + \gamma_{MN}\left(\frac{1}{6} - \frac{1}{2}R\right) \tag{18a}$$
$$G_{MI} = R_{MI} \tag{18b}$$
$$G_{EI} = R_{EI} + \gamma_{EI}\left(1 - \frac{1}{2}R\right) \tag{18c}$$

These equations can be derived from the variational principle with the Lagrangian $\mathscr{L} = \sqrt{\gamma}(R + 1)$. Adding a term with Lagrangian multipliers Y^{mn},

$$\mathscr{L}_1 = Y^{mn}_{(x)}(\gamma_{mn} - \bar{\gamma}_{mn})$$

where $\bar{\gamma}_{mn}$ are the expressions of the vertical components of the metric tensor expressed in the vertical coordinates of the corresponding fiber points of the group manifold, this results in the vanishing of G_{MN}. The purely vertical components of the field equations do not contribute. The metric on the fibers remains, $\bar{\gamma}$, as required. A generalization of this Lagrangian multiplier method is not used for the present purpose.

The expression of the curvature on G in terms of tensorial expressions on the base has formally already been performed in the last section. The use of this formalism for the field equations requires, however, consideration of the fact that the projection on the base manifold imposes on the horizontal line elements the unit of a length which is different from the cosmological unit. The present theory, as mentioned before, offers no explanation for the relative values of the two units of length. The result of their large ratio is that the torsion tensor (which describes the energy density of matter) in general assumes values much larger than the curvature related to it. We are, however, able *formally* to separate curvature from torsion in this theory and can use the Riemannian geometry for parallel transfer.

Expressed in terms of the curvature tensor H on the base, the horizontal components of the field equations in an orthonormal frame $\{e_J = \pi' A_J\}$ become

$$H^J{}_{EIJ} - \frac{3}{2} G H_{ABDE} H^{ABD}{}_I - K^J{}_{EI\|J} + K^J{}_{HI} K^H{}_{EJ} - T^H{}_{JI} K^J{}_{EH}$$
$$- \frac{1}{2}\gamma_{EI}\left(1 - \frac{3}{4} G H_{ABDJ} H^{ABDJ} + H^J{}_A{}^A{}_J + K^J{}_H{}^A K^H{}_{AJ} + T^{HA}{}_J K^J{}_{AH}\right) = 0 \tag{19}$$

where the terms quadratic in H must have the gravitational constant G (square of the Planck length) as factor. The double bar denotes the covariant derivative with respect to the complete connection Γ on the base (with the contortion term).

The decomposition (16) of the tensor H in B and Q results in cancellation of torsion terms, so that the horizontal component of the field equations is

$$B_{EI} - \frac{3}{2} G H_{ABDE} H^{ABD}{}_I - \frac{1}{2}\gamma_{EI}\left(B - \frac{3}{4} G H_{ABDJ} H^{ABDJ} + 1\right) = 0 \tag{19a}$$

Notice that the large value of the cosmological member shows that we are still using the cosmological unit of length in which G is extremely small (about 10^{-125}).

The horizontal equations contain the Einstein term with cosmological member and a term bilinear in the curvature tensor H with the gravitational constant G as a factor. The decomposition of H into B and Q allows one to separate in the bilinear term a pure torsion term without Riemannian curvature, a purely Riemannian term of vanishing trace, and a mixed term which is linear in B and Q. Compared with general relativity, the purely Riemannian term shows an additional energy-momentum term of the gravitational field.

This term is covariantly conserved only if the mixed horizontal–vertical equations have no torsion term (no source of the purely Riemannian part). This nonconservation of the purely gravitational tensor can be compared with the appearance of the Lorentz force in electrodynamics—as indeed in our gauge theory of geometry the mixed component can be called a gravitational analog of the Maxwell term with a source that may be interpreted as due to a Pauli-type interaction of the Riemannian tensor with matter spin. Apart from the pure torsion term with minimal coupling to the metric, which may be substituted for the matter tensor, we find also a mixed $B-Q$ part-interaction of metric curvature with torsion which results again from the Pauli term—an "interaction part" of the energy-momentum tensor. Notice that the trace of each of the mentioned terms contained in the expression bilinear in H vanishes separately. Then

$$B = -2, \qquad B^{h}{}_{j;h} = 0 \tag{20}$$

must thus be fulfilled for any solution in cosmological units. The remaining "right-hand side" to the Einstein term is of course also covariantly conserved and fulfills a conservation law together with an additional pseudo-tensor of the metric field. The purely metric term of the right-hand side is, however, as mentioned, not covariantly conserved if torsion does not vanish.

The mixed vertical–horizontal components of the field equations—the analog of the Maxwell or general Yang–Mills equations, are of the form

$$G_{MI} = R_{MI} = \frac{1}{2}\mathscr{C}_{MI}{}^{J}{}_{|J} + \frac{1}{4}\mathscr{C}_{M}{}^{K}{}_{J}T_{IK}{}^{J} = 0 \tag{21}$$

The derivative of $\mathscr{C}$, denoted by the bar—the horizontal derivative—becomes the invariant derivative of the Yang–Mills field on the base; the space-time components of this derivative are with the connection form $\bar{\omega}$ containing torsion. They are, according to our theorem, based on the Jacobi identities, canceled by the remaining Yang–Mills rotation, so that only the ordinary derivative remains, even on the base. This expression is, however, in our case of the form

$$D^{*}D\omega(A_J, A_I, A_J) \tag{21a}$$

which is well known, from the analog to Maxwell's equations, to be the metric (Riemannian) covariant divergence of a tensor density. The additional term containing torsion in (21) is cancelled in the formation of the total vertical derivative. We remain thus with the metric covariant divergence of the curvature form (without the contortion term in the covariant derivative). This form is still Lie algebra valued. We can write the equivalent tensor expression on the base. It becomes

$$H^{A}{}_{BI}{}^{J}{}_{;J} = 0 \tag{21b}$$

again with the metric covariant derivative of the metric g. The purely metric part of the expression (21b) is

$$B^{A}{}_{BI}{}^{J}{}_{;J} \tag{21c}$$

which we can recognize as the expression equivalent to the term suggested by Yang (1974) as the field equation of a gauge theory of gravitation.

We have alredy pointed out that it can be regarded as the Riemannian analog of Maxwell's equations—but in a gauge theory of $GL(4, r)$ it lacks the torsion part, which in the present theory is the term

$$Q^{A}{}_{BI}{}^{J}{}_{;J} \tag{21d}$$

which we interpret as the matter source of this equation. It forms in our theory an ingredient of the geometry of space-time and not a source of it, as in previous theories (Hehl *et al.*, 1976). Our invariance group indicates that orbits with vertical components on G are related to (inner) angular momentum, which suggests that the term (21c) is related to the spin tensor. It is equal to the term formed out of the metric g and must thus in general be small even if the torsion tensor itself is very large compared to the Riemannian curvature.

To check the correctness of our tensor formulation on the base, we consider the conservation law of the Einstein tensor in ten dimensions:

$$G_{R}{}^{S}{}_{;S} = 0 \tag{22}$$

the term $G_{N}{}^{S}{}_{;S}$ gives no significant contribution. The term $G_{E}{}^{R}{}_{;R}$ becomes after elimination of all terms which vanish in our geometry

$$G^{H}{}_{E|H} - G_{MI}\mathscr{C}^{MI}{}_{E} - G^{H}{}_{I}K^{I}{}_{EH} = 0 \tag{22a}$$

Written as tensorial expressions on the base, the first term is the covariant divergence of the tensor on the base corresponding to $G^{H}{}_{E}$. The third term cancels, however, the contortion part of this divergence, so that only the metric part remains. The second term vanishes because of the mixed component of the field equations. It is, however, instructive to separate the whole expression of (22a) into a complete torsion-free curvature part and a part containing torsion terms (part of them interacting with curvature).

The torsion-free part, unlike its purely Einsteinian term, is only then covariantly conserved if torsion vanishes. It is, as we know already from (19a), of the form

$$\left(B^{H}{}_{E} - \frac{1}{2}\delta^{H}_{E}B - \frac{3}{2}GB_{ABDE}B^{ABDH} - \frac{1}{2}\delta^{H}_{E} + \frac{3}{8}G\delta^{H}_{E}B_{ABDI}B^{ABDI}\right)_{;H} \tag{22b}$$

This divergence, due to the mixed field equations, vanishes only if the latter have no torsion part. If this is not true, only the divergence of the total expression is zero,

$$\left(B^{H}{}_{E} - \frac{1}{2}\delta^{H}_{E}B - \frac{3}{2}GH_{ABDE}H^{ABDH} - \frac{1}{2}\delta^{H}_{E} + \frac{3}{8}G\delta^{H}_{E}H_{ABDI}H^{ABDI}\right)_{;H} = 0 \tag{22c}$$

H contains the tensor Q, which occurs thus either bilinear or multiplied with the curvature tensor B as a nonminimal interaction term between metric and torsion.

4. CONCLUSIONS

We have contemplated evidence for the validity of the gravitational field equations in the very small and the very large. We concluded that we are lacking experimental evidence in the small, the description of the gravitational field as a self-interacting spin-two field being only based on extrapolative speculation. The description in the very large and for extremely strong gravitational fields we consider as inadequate because the classical equations are only compatible with the Schrödinger effect of elementary particle pair creation, but cannot by themselves describe it. The indication is thus very strong that also the much more significant virtual pair effects are not taken into account in the Einstein–Hilbert equations for extreme situations. We contemplate as candidates for a more adequate description in the large the equations of the author's gauge theory. The geometrical structure of this theory is particularly clear and simple because it is constructed on the group manifold of a semisimple group ($SO(3, 2)$) with its subgroup ($SO(3, 1)$) the proper Lorentz group as gauge group. The fiber bundle aspect as well as the Kaluza–Klein aspect of the theory are inherent in its structure—even the Einstein equations in ten dimensions result from the group geometry.

The physical interpretation of the geometrical structure is unusual. A connection is defined by assuming a horizontal vector space perpendicular to the vertical vectors. The Lie algebra-valued curvature two-form of this connection is interpreted as the gauge field; like every true Yang–Mills gauge field it is of spin one. The fact that our gauge group is an orthogonal subgroup of $GL(4, r)$ allows us, however, to separate this gauge field into a purely Riemannian part and a part formed out of the torsion two-form and its first derivatives. The former part is directly related to the metric gravitational field, which is assumed to have spin two. The latter part, which can in general not be assumed to vanish, has no analog in any existing theory and we tentatively relate it to the presence of nongravitational sources. The *homogeneous* Einstein equations on the total group manifold can thus be separated into a purely gravitational "left-hand member" and a "right-hand member" with torsion, assumed to be composed out of the nongravitational matter source and interaction terms of matter and gravitation which involve both the Riemannian curvature two-form and the torsion two-form. A constant of the dimension of a length (Planck length) is required for the correct dimension of the different terms on the base of the bundle (space-time). The purely vertical component of the equations is eliminated by Lagrange multipliers which achieve that the metric on the fibers remains for every admissible solution the same Cartan–Killing metric of the group manifold.

The purely horizontal component of the equation consists of the Einstein–Hilbert term of the metric on the base plus a term bilinear in the gauge field. The latter can again be decomposed into a term bilinear in the Riemann tensor—an additional energy-momentum tensor of the metric field—and a bilinear energy-momentum source term consisting of pure torsion terms and mixed interaction terms between torsion and the Riemannian curvature. All these bilinear terms have the square of the Planck length as factor.

The mixed vertical–horizontal part of the field equations are the equations of the gauge field. These can again be decomposed into a purely gravitational term which is equivalent to the expression suggested by Yang—and has therefore third derivatives of the metric—and a source term formed out of the torsion tensor and its first and second derivatives. If this source term does not vanish, the purely Riemannian part of the horizontal equations alone is not covariantly conserved.

It must be stressed that the bilinear energy-momentum source of the gauge field in the horizontal component of the total equations is of vanishing trace—as expected for a Yang–Mills field. If our conjecture is right that the torsion part represents a matter source, then we can conclude that the energy-momentum tensor of this matter has vanishing trace in this simple version of the theory. It is thus to be expected that it consists largely of matter with vanishing rest mass which can most effectively be produced by gravitational fields. An average matter source can even in this case hardly be expected to be of vanishing trace; to obtain a more realistic matter tensor one will in any case have to consider more sophisticated higher-dimensional versions of such a theory.

The points on the fiber are related to orthonormal frames and thus to angular position and systems of reference. The inner degrees of freedom are therefore related to (inner) angular momentum as an analog of charge in the original Kaluza–Klein theory. The possibility to convert spin into the dynamical variable of angular momentum is, besides the fact that the gauge field comprises the metric, a most peculiar feature of the present theory. A spinning test particle interacts clearly with the Riemannian curvature, yet it does not in general follow a geodesic of the metric γ on G (Halpern, 1992). The problem of spin motion will be treated from the present point of view in a broader context.

The vacuum solutions of general relativity are in general not torsion-free solutions of the present theory, but they are good approximations for the conditions of our solar system because the expression bilinear in the curvature tensor is there very small. An exception is the Schwarzschild–Weyl–Trefftz solution, which because of its spherical symmetry, like the anti-de Sitter metric, is an exact solution. Solutions of collapsing matter will of course behave very differently from the predictions of general relativity.

Although the present context does not yet yield a general realistic description of matter, the structure now exists to fulfill Einstein's requirement to geometrize the right-hand member of his field equations; this has in principle here been achieved, at the price of relaxing Einstein's other requirement of minimal coupling of the metric to matter. The latter requirement has, however, in any case to be relaxed to describe the Schrödinger effect and virtual pair effects realistically. Somehow, like for example, the principle of the conservation of mass, the requirement in question has to be generalized to meet the demands of modern physics—in spite of its usefulness (and even necessity) for the evolution of physics in the past. The relaxation of the requirement entails, however, modified field equations with higher than second derivatives of the metric. The structure of the present equations with the inclusion of the third-order Yang term may become a more lasting feature of gravitational theory. If one believes that quantization is closely related to gauge principles, the present gauge theory with a spin-one Yang–Mills field and its master equation, containing gravitation and matter, opens up a new outlook to the long-stagnating attempts to quantize the gravitational field separately from matter.

There is little doubt that in future developments of the present approach a more general non-Riemannian geometry than the restricted present one will have to be used. The present theory can be termed a gauge theory of the restricted non-Riemannian geometry.

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